Cuaderno de notas de trabajo

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Cuaderno Sn 2

CAMBIO DE VARIABLE EN LAS ECUACIONES DIFERENCIALES DE SENDEROS.

1) Ecuaciones diferenciales de los senderos:

$$\Delta_{ip} \frac{d^2 x^i}{ds^2} = \left(\frac{\partial h_{pj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^p} \right) \frac{dx^j}{ds} \frac{dx^k}{ds};$$

2) Introducción de la nueva variable σ definida por

$$d\sigma^2 = g_{ij}\, dx^i\, dx^j;$$

3)

$$ds^2 = \Delta_{ij}\, dx^i\, dx^j.$$

$$\boxed{\boxed{\frac{ds}{d\sigma}}}$$

$$ds^2 = \Delta_{ij}\, dx^i\, dx^j$$

$$\left(\frac{ds}{d\sigma}\right)^2 = \Delta_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}$$

$$\boxed{\boxed{\frac{ds}{d\sigma} = \sqrt{\Delta_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}}}}$$

$$\boxed{\frac{dx^p}{ds}}$$

$$\frac{dx^p}{ds} = \frac{\left(\frac{dx^p}{d\sigma}\right)}{\left(\frac{ds}{d\sigma}\right)} = \frac{\frac{dx^p}{d\sigma}}{\sqrt{\Delta_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}}$$

$$\boxed{\frac{dx^p}{ds} = \frac{\frac{dx^p}{d\sigma}}{\sqrt{\Delta_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}}}$$

$$\boxed{\frac{dx^p}{ds}\frac{dx^q}{ds}}$$

$$\boxed{\frac{dx^p}{ds}\frac{dx^q}{ds} = \frac{\frac{dx^p}{d\sigma}\frac{dx^q}{d\sigma}}{\Delta_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}}$$

Operador: $\boxed{\dfrac{d}{ds}}$

$$\frac{d}{ds} = \frac{1}{\left(\frac{ds}{d\sigma}\right)} \frac{d}{d\sigma}$$

$$\boxed{\frac{d}{ds} = \frac{1}{\sqrt{\Delta_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}}} \frac{d}{d\sigma}}$$

$$\boxed{\frac{d^2 x^i}{ds^2}}$$

$$\frac{dx^i}{ds} = \frac{1}{\sqrt{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}}\frac{dx^i}{d\sigma}$$

$$\frac{d^2x^i}{ds^2} = \frac{1}{\sqrt{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}}\left\{\frac{1}{\sqrt{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}}\frac{d^2x^i}{d\sigma^2} - \frac{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{d^2x^b}{d\sigma^2}}{\left\{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right\}^{3/2}}\frac{dx^i}{d\sigma}\right\}$$

$$\boxed{\frac{d^2x^i}{ds^2} = \left[\frac{1}{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}}\frac{d^2x^i}{d\sigma^2} - \frac{1}{\left\{\Delta_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right\}^2}\frac{dx^i}{d\sigma}\Delta_{cd}\frac{dx^c}{d\sigma}\frac{d^2x^d}{d\sigma^2}\right]}$$

Notación:

$$\boxed{\varphi^i = \frac{dx^i}{d\sigma}}$$

$$\boxed{\alpha^i = \frac{d^2x^i}{d\sigma^2}}$$

$$\boxed{\alpha_p = \Delta_{pi}\frac{d^2x^i}{d\sigma^2}} \qquad \boxed{\varphi_p = \Delta_{pi}\frac{dx^i}{d\sigma}}$$

$$\varphi_a\varphi^a = \varphi^2 = \hat{\varphi}^2$$

Cuadrivector $\hat{\varphi}$ $\quad \varphi^i$

Cuadrivector $\hat{\alpha}$ $\quad \alpha^i$

$$\frac{dx^p}{ds}\frac{dx^q}{ds} = \frac{\varphi^p \varphi^q}{\Delta_{ij}\varphi^i \varphi^j}$$

$$\frac{d^2x^i}{ds^2} = \frac{1}{\Delta_{ab}\varphi^a\varphi^b}\alpha^i - \frac{1}{\{\Delta_{ab}\varphi^a\varphi^b\}^2}\varphi^i\varphi_d\alpha^d$$

Senderos:

$$\frac{1}{\varphi_a\varphi^a}\alpha_p - \frac{1}{(\varphi_a\varphi^a)^2}\varphi_p\left(\varphi_d\alpha^d\right) =$$

$$\left(\frac{\partial h_{pj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^p}\right)\frac{\varphi^j\varphi^k}{(\varphi_a\varphi^a)}$$

$$\boxed{\hat{\varphi}^2\alpha_p - \left(\hat{\varphi}\cdot\hat{\alpha}\right)\varphi_p = \hat{\varphi}^2\left(\frac{\partial h_{pj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^p}\right)\varphi^j\varphi^k}$$

Ecuación Diferencial de los Senderos.

Variable : σ ; $d\sigma^2 = g_{pq}\, dx^p dx^q$

Cuadrivectores: $\hat{\varphi}$ y $\hat{\alpha}$.

$$\varphi^i = \frac{dx^i}{d\sigma} \; ; \quad \alpha^i = \frac{d\varphi^i}{d\sigma} = \frac{d^2x^i}{d\sigma^2}.$$

$$\hat{\varphi}^2 = \varphi_i \varphi^i \; ; \quad \hat{\varphi}\cdot\hat{\alpha} = \varphi_i \alpha^i .$$

$$\boxed{\hat{\varphi}^2 \alpha_p - (\hat{\varphi}\cdot\hat{\alpha})\varphi_p = \hat{\varphi}^2\left(\frac{\partial h_{pj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^p}\right)\varphi^j \varphi^k}$$

Senderos

Consideraciones sobre $\hat{\varphi} \cdot \hat{\alpha}$.

$$\frac{dx^p}{ds} = \frac{\frac{dx^p}{d\sigma}}{\sqrt{\Delta_{ab} \frac{dx^a}{d\sigma} \frac{dx^b}{d\sigma}}} = \frac{\varphi^p}{\sqrt{\Delta_{ab}\, \varphi^a \varphi^b}} = \frac{\varphi^p}{\varphi}$$

$$\boxed{\frac{dx^p}{ds} = \frac{\varphi^p}{\varphi}} \qquad \varphi = \sqrt{\hat{\varphi}^2}$$

$$\boxed{\frac{d^2x^i}{ds^2} = \frac{1}{\hat{\varphi}^2}\alpha^i - \frac{1}{\hat{\varphi}^4}\varphi^i \left(\hat{\varphi} \cdot \hat{\alpha}\right)}$$

$$\Delta_{pi} \frac{dx^p}{ds} \frac{d^2x^i}{ds^2} = \frac{(\hat{\varphi} \cdot \alpha)}{\varphi^3} - \frac{\varphi^2}{\varphi^5} (\hat{\varphi} \cdot \hat{\alpha}) \equiv 0$$

Transformación de las derivadas con respecto a σ en derivadas con respecto a s.

$$\boxed{\boxed{d\sigma^2 = g_{ij}\, dx^i\, dx^j}}$$

$$d\sigma = \sqrt{g_{ij}\, dx^i\, dx^j}$$

$$\frac{d}{d\sigma} = \frac{ds}{\sqrt{g_{ij}\, dx^i\, dx^j}}\, \frac{d}{ds}$$

$$\boxed{\boxed{\frac{d}{d\sigma} = \frac{1}{\sqrt{g_{ij}\, \frac{dx^i}{ds}\, \frac{dx^j}{ds}}}\, \frac{d}{ds}}}$$

Transformación de la segunda derivada con respecto a σ

$$\frac{d^2}{d\sigma^2} = \frac{1}{\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}\frac{d}{ds}\left\{\frac{1}{\sqrt{g_{ij}\frac{dx^0}{ds}\frac{dx^j}{ds}}}\frac{d}{ds}\right\}$$

$$\frac{d^2}{d\sigma^2} = \frac{1}{\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}\left[\frac{1}{\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}\frac{d^2}{ds^2} - \frac{1}{2}\frac{\left\{\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 2g_{ij}\frac{d^2x^i}{ds^2}\frac{dx^j}{ds}\right\}}{\left\{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right\}^{3/2}}\frac{d}{ds}\right]$$

$$\boxed{\frac{d^2}{d\sigma^2} = \frac{1}{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\frac{d^2}{ds^2} - \frac{1}{2}\frac{\frac{\partial g_{pq}}{\partial x^r}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^r}{ds}}{\left\{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right\}^2}\frac{d}{ds} - \frac{g_{pq}\frac{d^2x^p}{ds^2}\frac{dx^q}{ds}}{\left\{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right\}^2}\frac{d}{ds}}$$

$$\boxed{\frac{d}{d\sigma} = \frac{1}{\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}\frac{d}{ds}}$$

Transformación de derivadas con respecto a σ

— primeras y segundas — en derivadas con repecto a s.

Ecuación de las Geodésicas usando el parámetro τ.

1) espacio de Riemann

$$ds^2 = g_{ij}\, dx^i dx^j.$$

2) Parámetro τ:

$$\frac{ds^2}{d\tau^2} = \left(\frac{ds}{d\tau}\right)^2 = T(\tau).$$

3) Principio variacional

$$0 = \delta \int_A^B \left\{ \sqrt{g_{ij}\, dx^i dx^j} + \lambda \left\{ \frac{g_{pq}\, dx^p dx^q}{d\tau^2} - T(\tau) \right\} \right\}$$

$$d\bar{W} = \sqrt{g_{ij}\,dx^i dx^j} + \lambda\left\{ \frac{g_{pq}\,dx^p dx^q}{d\tau} - T\,d\tau \right\}$$

$$\cancel{\delta(d\bar{W}) = g_{ij}\,dx^i\,dt}$$

$$\delta(d\bar{W}) = \left[\frac{\frac{\partial g_{ij}}{\partial x^k}\delta x^k dx^i dx^j}{2\sqrt{g_{uv}\,dx^u dx^v}} + \frac{g_{ij}\,dx^i d(\delta x^j)}{\sqrt{g_{uv}\,dx^u dx^v}} + \lambda \frac{\partial g_{pq}}{\partial x^r}\delta x^r \frac{dx^p}{d\tau} dx^q + 2\lambda g_{pq}\frac{dx^p}{d\tau} d(\delta x^q) \right]$$

$$\boxed{(\delta J)_1 = \int_A^B \left[\frac{\frac{\partial g_{ij}}{\partial x^k}\delta x^k \frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}{2\sqrt{g_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}}}\,d\tau + \lambda \frac{\partial g_{pq}}{\partial x^k}\delta x^k \frac{dx^p}{d\tau}\frac{dx^q}{d\tau} \right] d\tau}$$

$$d\bar{W} = \sqrt{g_{ij}\, dx^i dx^j} + \lambda \left\{ g_{pq} \frac{dx^p dx^q}{d\tau} - T \frac{\xi}{d\tau} \right\}$$

$$\cancel{\delta(d\bar{W}) = g_{ij}\, dx^i dt}$$

$$\delta(d\bar{W}) = \left[\frac{\frac{\partial g_{ij}}{\partial x^k} \delta x^k dx^i dx^j}{2\sqrt{g_{uv}\, dx^u dx^v}} + \frac{g_{ij}\, dx^i d(\delta x^j)}{\sqrt{g_{uv}\, dx^u dx^v}} \right]$$

$$\boxed{(\delta J)_2 = \int_A^B \frac{g_{ij} \frac{dx^i}{d\tau}}{\sqrt{g_{uv} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau}}} \, d(\delta x^j)}$$

$$(\delta_x J)_2 = \int_A^B \Bigg\{ - \frac{\frac{\partial g_{ij}}{\partial x^k} \frac{dx^k}{d\tau} \frac{dx^i}{d\tau}}{\sqrt{g_{uv} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau}}} \delta x^j \, d\tau - \frac{g_{ij} \frac{d^2 x^i}{d\tau^2} \delta x^j}{\sqrt{g_{uv} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau}}} d\tau$$

$$+ \frac{g_{ij} \frac{dx^i}{d\tau} \delta x^j}{2 \left\{ g_{uv} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau} \right\}^{3/2}} \cdot \frac{\partial g_{uv}}{\partial x^w} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau} \frac{dx^w}{d\tau} d\tau + \frac{g_{ij} \frac{dx^i}{d\tau} \delta x^j}{\left\{ g_{uv} \frac{dx^u}{d\tau} \frac{dx^v}{d\tau} \right\}^{3/2}} g_{uv} \frac{dx^u}{d\tau} \frac{d^2 x^v}{d\tau^2} d\tau \Bigg\}$$

$$\delta I_2 = \int_A^B \left\{ -\frac{\frac{\partial g_{ik}}{\partial x^j}\frac{dx^j}{d\tau}\frac{dx^i}{d\tau}}{\sqrt{g_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}}}\delta x^k d\tau - \frac{g_{ik}\frac{d^2x^i}{d\tau^2}}{\sqrt{g_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}}}\delta x^k d\tau \right.$$

$$+ \frac{g_{ik}\frac{dx^i}{d\tau}}{2\left\{g_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\right\}^{3/2}}\frac{\partial g_{uv}}{\partial x^w}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\cancel{\frac{dx^w}{d\tau}}\delta x^k d\tau$$

$$\left. + \frac{g_{ik}\frac{dx^i}{d\tau}}{\cancel{2}\left\{g_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\right\}^{3/2}} g_{uv}\frac{dx^u}{d\tau}\frac{d^2x^v}{d\tau^2}\delta x^k d\tau \right\}$$

$$\boxed{\left(\delta J\right)_3 = 2\lambda \int_A^B \left\{ g_{pq} \frac{dx^p}{d\tau}\, d(\delta x^q) \right\}}$$

$$\boxed{\delta J_3 = 2\lambda \int_A^B \left\{ \begin{array}{l} -\dfrac{\partial g_{pk}}{\partial x^q} \dfrac{dx^p}{d\tau} \dfrac{dx^q}{d\tau}\, \delta x^k\, d\tau \\[2ex] -\, g_{pk} \dfrac{d^2 x^p}{d\tau^2}\, \delta x^k\, \delta\tau \end{array} \right\}}$$

Geodésicas

Términos que contienen a las segundas derivadas de las coordenadas con respecto a τ

$$-\frac{ds^2}{d\tau^2}\, g_{ik} \frac{d^2x^i}{d\tau^2} + \cancel{\tfrac{1}{2}}\, g_{ik} \frac{dx^i}{d\tau}\, g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d^2x^\nu}{d\tau}$$

$$-2\lambda \left(\frac{ds}{d\tau}\right)^3 g_{pk} \frac{d^2x^p}{d\tau^2}$$

Geodésicas

Otros Términos (ver 102!)

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 \frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + \lambda\left(\frac{ds}{d\tau}\right)^3 \frac{\partial g_{pq}}{\partial x^k}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau} \\
&- \left(\frac{ds}{d\tau}\right)^2 \frac{\partial g_{ik}}{\partial x^j}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + \frac{1}{2} g_{ik}\frac{dx^i}{d\tau}\frac{\partial g_{uv}}{\partial x^w}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\frac{dx^w}{d\tau} \\
&- 2\lambda\left(\frac{ds}{d\tau}\right)^3 \frac{\partial g_{pk}}{\partial x^q}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}
\end{aligned}
$$

Geodésicas.

$$
\left.\begin{aligned}
&-\left(\frac{ds}{d\tau}\right)^2\left[g_{ik}\frac{d^2x^i}{d\tau^2}+\frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right\}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\\
&+\cancel{\tfrac{1}{2}}\,g_{ik}\frac{dx^i}{d\tau}g_{uv}\frac{dx^u}{d\tau}\left[\frac{d^2x^v}{d\tau^2}\right]\cancel{4}\,\frac{1}{2}\\
&+\frac{1}{2}g_{ik}\frac{dx^i}{d\tau}\left\{\frac{\partial g_{uw}}{\partial x^v}+\frac{\partial g_{vw}}{\partial x^u}-\frac{\partial g_{uv}}{\partial x^w}\right\}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\frac{dx^w}{d\tau}\\
&+\lambda\left[\begin{aligned}&-2\left(\frac{ds}{d\tau}\right)^3 g_{pk}\frac{d^2x^p}{d\tau^2}+\left(\frac{ds}{d\tau}\right)^3\frac{\partial g_{pq}}{\partial x^k}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\\
&-2\left(\frac{ds}{d\tau}\right)^3\frac{\partial g_{pk}}{\partial x^q}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\end{aligned}\right]
\end{aligned}\right\}=0
$$

Geodésicas.

$$
\left.\begin{aligned}
&-\left(\frac{ds}{d\tau}\right)^2\left[g_{ik}\frac{d^2x^i}{d\tau^2}+\Gamma_{ij,k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\\
&+\cancel{\tfrac{1}{2}}\,g_{ik}\frac{dx^i}{d\tau}g_{uv}\frac{dx^u}{d\tau}\frac{d^2x^v}{d\tau^2}\\
&+\cancel{2}\,g_{ik}\frac{dx^i}{d\tau}\Gamma_{uv,w}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\frac{dx^w}{d\tau}\\
&+\lambda\left(\frac{ds}{d\tau}\right)^3\left[-2g_{ik}\frac{d^2x^i}{d\tau^2}-\Gamma_{pq,k}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right]
\end{aligned}\right\}=0
$$

Geodésicas.

Introducción de los símbolos de Christoffel de Segunda Especie.

$$-\left(\frac{ds}{d\tau}\right)^2 g_{ik}\left[\frac{d^2x^i}{d\tau^2}+\Gamma^i_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\right]$$

$$+\lambda\left(\frac{ds}{d\tau}\right)^3 g_{ik}\left[\frac{d^2x^i}{d\tau^2}+\Gamma^i_{uv}\frac{dx^u}{d\tau}\frac{dx^v}{d\tau}\right]$$

$$+g_{ik}\frac{dx^i}{d\tau}g_{uv}\frac{dx^u}{d\tau}\frac{d^2x^v}{d\tau^2}+\cancel{2}g_{ik}\frac{dx^i}{d\tau}g_{wv}\Gamma^v_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\frac{dx^w}{d\tau}$$

Geodésicas

Geodésicas

$$-\left(\frac{ds}{d\tau}\right)^2 g_{ik}\left[\frac{d^2x^i}{d\tau^2}+\Gamma^i_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right]$$

$$-\lambda\left(\frac{ds}{d\tau}\right)^3 g_{ik}\left[\frac{d^2x^i}{d\tau^2}+\Gamma^i_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right]$$

$$+\,g_{ik}\frac{dx^i}{d\tau}\,g_{uv}\frac{dx^u}{d\tau}\left[\frac{d^2x^v}{d\tau^2}+\Gamma^v_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right]$$

GEODESICAS.

$$\left[-\left(\frac{ds}{d\tau}\right)^2 g_{rk} - \lambda\left(\frac{ds}{d\tau}\right)^3 g_{rk} + g_{ik}\frac{dx^i}{d\tau} g_{vr}\frac{dx^v}{d\tau}\right]\left[\frac{d^2x^r}{d\tau^2} + \Gamma^r_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right] = 0$$

$$\boxed{\text{Vector: } V^r = \frac{d^2x^r}{d\tau^2} + \Gamma^r_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}}$$

$$\boxed{\left[-T^2 g_{rk} - \lambda T^3 g_{kr} + g_{ik}\frac{dx^i}{d\tau} g_{vr}\frac{dx^v}{d\tau}\right] V^r = 0}$$

$$-T^2 V_k - \lambda T^3 V_k + g_{ik}\frac{dx^i}{d\tau} V_v \frac{dx^v}{d\tau} = 0$$

$$\boxed{-\left(T^2 + \lambda T^3\right)\hat{V} + \frac{d\hat{r}}{d\tau}\left(\hat{V}\cdot\frac{d\hat{r}}{d\tau}\right) = 0}$$

$$\left(\frac{d\hat{r}}{d\tau}\right)^2 = g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = T^2$$

$$\cdot\ \frac{d\hat{r}}{d\tau}$$

$$\left[\cancel{-T^2} - \lambda T^3 + \cancel{T^2}\right]\left(\hat{V}\cdot\frac{d\hat{r}}{d\tau}\right) = 0$$

$$\hat{V}\cdot\frac{d\hat{r}}{d\tau} = 0$$

Espacio: $x^1, x^2, x^3, \cdots x^n$.

1) Principio Variacional.

$$\boxed{\delta \int_A^B \frac{dl^2}{dm} = 0}$$

2) Usese m como variable, ~~independiente~~.

3) $$\boxed{dl^2 = \lambda_{ij}\, dx^i dx^j}$$

4) $$\boxed{\Lambda_{ijk} = \frac{1}{2}\left[\frac{\partial \lambda_{ki}}{\partial x^j} + \frac{\partial \lambda_{jk}}{\partial x^i} - \frac{\partial \lambda_{ij}}{\partial x^k}\right]}.$$

5) $$\boxed{dm^2 = \sqrt{\mu_{pq}\, dx^p\, dx^q}}$$

6) $$\boxed{M_{ijk} = \frac{1}{2}\left[\frac{\partial \mu_{ki}}{\partial x^j} + \frac{\partial \mu_{jk}}{\partial x^i} - \frac{\partial \mu_{ij}}{\partial x^k}\right]}$$

$$\boxed{dl^2 = \lambda_{ij}\, dx^i dx^j}$$

$$\boxed{\delta(dl^2) = \frac{\partial \lambda_{ij}}{\partial x^\alpha}\delta x^\alpha dx^i dx^j + 2\,\lambda_{i\alpha}\, dx^i d(\delta x^\alpha)}$$

$$\boxed{dm = \sqrt{\mu_{pq}\, dx^p\, dx^q}}$$

$$\delta(dm) = \frac{1}{dm}\left[\frac{1}{2}\frac{\partial \mu_{pq}}{\partial x^\alpha}\delta x^\alpha\, dx^p\, dx^q + \mu_{p\alpha}\, dx^p\, d(\delta x^\alpha)\right]$$

$$\boxed{\delta(dm) = \frac{1}{2}\frac{\partial \mu_{pq}}{\partial x^\alpha}\delta x^\alpha \frac{dx^p}{dm} dx^q + \mu_{p\alpha}\frac{dx^p}{dm} d(\delta x^\alpha)}$$

$$\boxed{\delta \frac{dl^2}{dm}}$$

$$\cancel{\frac{\partial \lambda_{ij}}{\partial x^\alpha} dx^i dx^j \delta x^\alpha}$$

$$\boxed{\begin{aligned} &\frac{\partial \lambda_{ij}}{\partial x^\alpha} \delta x^\alpha \frac{dx^i}{dm} \frac{dx^j}{dm} dm + 2 \lambda_{i\alpha} \frac{dx^i}{dm} d(\delta x^\alpha) \\ &- \frac{1}{2} \lambda_{ij} \frac{\partial \mu_{pq}}{\partial x^\alpha} \delta x^\alpha \frac{dx^i}{dm} \frac{dx^j}{dm} \frac{dx^p}{dm} \frac{dx^q}{dm} dm \\ &- \lambda_{ij} \mu_{p\alpha} \frac{dx^i}{dm} \frac{dx^j}{dm} \frac{dx^p}{dm} d(\delta x^\alpha) \end{aligned}}$$

$$\boxed{+2\int_A \lambda_{i\alpha} \frac{dx^i}{dm}\, d(\delta x^\alpha)}$$

$$\boxed{-2 \frac{\partial \lambda_{i\alpha}}{\partial x^j} \frac{dx^i}{dm} \frac{dx^j}{dm} \delta x^\alpha \, dm - 2 \lambda_{\alpha\beta} \frac{d^2 x^\beta}{dm^2} \delta x^\alpha \, dm}$$

$$-\cancel{2}\int_A^B \lambda_{ij}\,\Gamma^{\mu}_{p\alpha}\frac{dx^i}{dm}\frac{dx^j}{dm}\frac{dx^p}{dm}\,d(\delta x^\alpha)$$

$$+\frac{\partial\lambda_{ij}}{\partial x^q}\Gamma_{\mu p\alpha}\frac{dx^i}{dm}\frac{dx^j}{dm}\frac{dx^p}{dm}\frac{dx^q}{dm}\,\delta x^\alpha\,dm$$

$$+\lambda_{ij}\frac{\partial\mu_{p\alpha}}{\partial x^q}\frac{dx^i}{dm}\frac{dx^j}{dm}\frac{dx^p}{dm}\frac{dx^q}{dm}\,\delta x^\alpha\,dm$$

$$+2\lambda_{i\beta}\,\Gamma_{\mu p\alpha}\frac{dx^i}{dm}\frac{d^2x^\beta}{dm^2}\frac{dx^p}{dm}\,\delta x^\alpha\,dm$$

$$+\lambda_{ij}\,\Gamma_{\mu\alpha\beta}\frac{dx^i}{dm}\frac{dx^j}{dm}\frac{d^2x^\beta}{dm^2}\,\delta x^\alpha\,dm$$

EXTREMALES. TÉRMINOS QUE
CONTIENEN A LAS SEGUNDAS
DERIVADAS DE LAS COORDENADAS
CON RESPECTO A $\underline{\underline{m}}$

EXTREMALES. TÉRMINOS
QUE CONTIENEN A LAS
PARCIALES DEL TENSOR
λ.

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$$\frac{\partial \lambda_{ij}}{\partial x^\alpha} \frac{dx^i}{dm} \frac{dx^j}{dm}$$

$$-2 \frac{\partial \lambda_{i\alpha}}{\partial x^j} \frac{dx^i}{dm} \frac{dx^j}{dm}$$

$$+ \frac{\partial \lambda_{ij}}{\partial x^q} \mu_{p\alpha} \frac{dx^i}{dm} \frac{dx^j}{dm} \frac{dx^p}{dm} \frac{dx^q}{dm}$$

$$-2 \Lambda_{ij,\alpha} \frac{dx^i}{dm} \frac{dx^j}{dm}$$

$$+ 2 \mu_{\alpha p} \Lambda_{uv,w} \frac{dx^p}{dm} \frac{dx^u}{dm} \frac{dx^v}{dm} \frac{dx^w}{dm}$$

EXTREMALES. TÉRMINOS QUE CONTIENEN A LAS PARCIALES DEL TENSOR C^{μ}.

$$- \frac{1}{2}\left(\frac{dl}{dm}\right)^2 \frac{\partial \mu_{pq}}{\partial x^\alpha} \frac{dx^p}{dm} \frac{dx^q}{dm}$$

$$+ \left(\frac{dl}{dm}\right)^2 \frac{\partial \mu_{p\alpha}}{\partial x^q} \frac{dx^p}{dm} \frac{dx^q}{dm}$$

$$\boxed{\left(\frac{dl}{dm}\right)^2 \mathrm{M}_{pq,\alpha} \frac{dx^p dx^q}{dm\, dm}}$$

EXTREMALES.

$$\left\{\begin{aligned}
&-2\left[\lambda_{\alpha\beta}\frac{d^2x^\beta}{dm^2} + \Lambda_{ij,\alpha}\frac{dx^i}{dm}\frac{dx^j}{dm}\right] \\
&+2\mu_{\alpha p}\frac{dx^p}{dm}\frac{dx^q}{dm}\left[\lambda_{\beta q}\frac{d^2x^\beta}{dm^2} + \Lambda_{ij,q}\frac{dx^i}{dm}\frac{dx^j}{dm}\right] \\
&+\left(\frac{dl}{dm}\right)^2\left[\mu_{\alpha\beta}\frac{d^2x^\beta}{dm^2} + M_{ij,\alpha}\frac{dx^i}{dm}\frac{dx^j}{dm}\right]
\end{aligned}\right\} = 0$$

EXTREMALES.

$$\boxed{\left.\begin{aligned}&-2\left\{\begin{matrix}\Lambda\\ m\end{matrix}\right\}_{\alpha}+2\mu_{\alpha p}\frac{dx^{p}}{dm}\frac{dx^{q}}{dm}\left\{\begin{matrix}\Lambda\\ m\end{matrix}\right\}_{q}\\&+\left(\frac{dl}{dm}\right)^{2}\left\{\begin{matrix}M\\ m\end{matrix}\right\}_{\alpha}\end{aligned}\right\}=0}$$

CAMBIO DE VARIABLE.

$$m \longrightarrow s.$$

$$\boxed{ds^2 = g_{ij}\, dx^i\, dx^j}$$

$$\boxed{dm^2 = \mu_{ij}\, dx^i\, dx^j}$$

$$\boxed{\left(\frac{dm}{ds}\right)^2 = \mu_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}}$$

$$\frac{d}{dm} = \frac{ds}{dm} \frac{d}{ds}$$

$$\boxed{\frac{d}{dm} = \left(\frac{dm}{ds}\right)^{-1} \frac{d}{ds}}$$

$$\boxed{\frac{dx^i}{dm} = \left(\frac{dm}{ds}\right)^{-1} \frac{dx^i}{ds}}$$

$$\boxed{\frac{dx^i}{dm} = \left(\frac{dm}{ds}\right)^{-1} \frac{dx^i}{ds}}$$

$$\frac{d^2x^i}{dm^2} = \left(\frac{dm}{ds}\right)^{-1} \left[-\left(\frac{dm}{ds}\right)^{-2} \frac{d^2m}{ds^2} \frac{dx^i}{ds} + \left(\frac{dm}{ds}\right)^{-1} \frac{d^2x^i}{ds^2} \right]$$

$$\boxed{\frac{d^2x^i}{dm^2} = \left[-\left(\frac{dm}{ds}\right)^{-3} \frac{d^2m}{ds^2} \frac{dx^i}{ds} + \left(\frac{dm}{ds}\right)^{-2} \frac{d^2x^i}{ds^2} \right]}$$

$$\cancel{\frac{d^2m}{ds^2} = \frac{1}{2}\left(\frac{dm}{ds}\right)^{-1}\frac{dx^k}{ds}}$$

$$\frac{d^2m}{ds^2} = \frac{\frac{\partial\mu_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 2\mu_{ij}\frac{dx^i}{ds}\frac{d^2x^j}{ds^2}}{2\sqrt{\mu_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}$$

$$\frac{d^2m}{ds^2} = \left[\frac{1}{2}\frac{\partial\mu_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{dm}{ds}\right)^{-1} + \mu_{ij}\left(\frac{dm}{ds}\right)^{-1}\frac{dx^i}{ds}\frac{d^2x^j}{ds^2}\right]$$

$$\cancel{\frac{d^2 m}{ds^2} = M_{ijk}}$$

$$\boxed{\frac{d^2 m}{ds^2} = \left[\left(\frac{dm}{ds}\right)^{-1} M_{ijk}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds}\,\frac{dx^k}{ds} + \left(\frac{dm}{ds}\right)^{-1} \mu_{ij}\,\frac{dx^i}{ds}\,\frac{d^2 x^j}{ds^2}\right]}$$

$$\boxed{\frac{d^2 m}{ds^2} = \left(\frac{dm}{ds}\right)^{-1} \frac{dx^i}{ds}\left[\mu_{ij}\,\frac{d^2 x^j}{ds^2} + M_{pq,i}\,\frac{dx^p}{ds}\,\frac{dx^q}{ds}\right]}$$

$$\boxed{\frac{d^2 m}{ds^2} = \left(\frac{dm}{ds}\right)^{-1} \frac{dx^i}{ds}\left[\frac{M}{s}\right]_i}$$

$$\frac{d^2x^i}{dm^2} = \left[-\left(\frac{dm}{ds}\right)^{-3} \frac{dx^i}{ds} \left[\left(\frac{dm}{ds}\right)^{-1} \frac{dx^r}{ds} \left\langle \mu_{rn} \frac{d^2x^n}{ds^2} + M_{pq,r} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\rangle \right] + \left(\frac{dm}{ds}\right)^{-2} \frac{d^2x^i}{ds^2} \right]$$

$$\boxed{\frac{d^2x^i}{dm^2} = -\left(\frac{dm}{ds}\right)^{-4} \frac{dx^i}{ds} \frac{dx^r}{ds} \begin{bmatrix} M \\ s \end{bmatrix}_r + \left(\frac{dm}{ds}\right)^{-2} \frac{d^2x^i}{ds^2}}$$

$$\boxed{\left[\frac{\Lambda}{m}\right]_\alpha = \lambda_{\alpha a}\frac{d^2x^a}{dm^2} + \Lambda_{pq,\alpha}\frac{dx^p}{dm}\frac{dx^q}{dm}}$$

$$\left[\frac{\Lambda}{m}\right]_\alpha = \left\{ \begin{aligned} &\lambda_{\alpha a}\left[-\left(\frac{dm}{ds}\right)^{-4}\frac{dx^a}{ds}\frac{dx^r}{ds}\left[\frac{M}{s}\right]_r + \left(\frac{dm}{ds}\right)^{-2}\frac{d^2x^a}{ds^2}\right] \\ &\cancel{+\lambda_{\alpha a}\left(\frac{dm}{ds}\right)^{-2}\frac{d^2x^a}{ds^2}} \\ &+ \Lambda_{pq,\alpha}\frac{dx^p}{ds}\frac{dx^q}{ds}\left(\frac{dm}{ds}\right)^{-2} \end{aligned} \right\}$$

$$\begin{bmatrix}\Lambda \\ m\end{bmatrix}_\alpha = \left[-\left(\frac{dm}{ds}\right)^{-4} \lambda_{\alpha a} \frac{dx^a}{ds}\frac{dx^r}{ds}\begin{bmatrix}M \\ s\end{bmatrix}_r + \left(\frac{dm}{ds}\right)^{-2}\left[\lambda_{\alpha a}\frac{d^2x^a}{ds^2} + \Lambda_{pq,\alpha}\frac{dx^p}{ds}\frac{dx^q}{ds}\right]\right]$$

$$\boxed{\begin{bmatrix}\Lambda \\ m\end{bmatrix}_\alpha = -\left(\frac{dm}{ds}\right)^{-4} \lambda_{\alpha a} \frac{dx^a}{ds}\frac{dx^r}{ds}\begin{bmatrix}M \\ s\end{bmatrix}_r + \left(\frac{dm}{ds}\right)^{-2}\begin{bmatrix}\Lambda \\ s\end{bmatrix}_\alpha}$$

EXTREMALES

$$
\left.\begin{aligned}
&+2\left(\frac{dm}{ds}\right)^{-4}\lambda_{\alpha a}\frac{dx^a}{ds}\frac{dx^r}{ds}\begin{bmatrix} M \\ s \end{bmatrix}_r - 2\left(\frac{dm}{ds}\right)^{-2}\begin{bmatrix} \Lambda \\ s \end{bmatrix}_\alpha \\
&+2\mu_{\alpha p}\frac{dx^p}{ds}\frac{dx^q}{ds}\left(\frac{dm}{ds}\right)^{-2}\left[\begin{aligned}&-\left(\frac{dm}{ds}\right)^{-4}\lambda_{qa}\frac{dx^a}{ds}\frac{dx^r}{ds}\begin{bmatrix} M \\ s \end{bmatrix}_r \\ &+\left(\frac{dm}{ds}\right)^{2}\begin{bmatrix} \Lambda \\ s \end{bmatrix}_q\end{aligned}\right] \\
&+\left(\frac{dl}{ds}\right)^{2}\left(\frac{dm}{ds}\right)^{-2}\left[\begin{aligned}&-\left(\frac{dm}{ds}\right)^{-4}\mu_{\alpha a}\frac{dx^a}{ds}\frac{dx^r}{ds}\begin{bmatrix} M \\ s \end{bmatrix}_r \\ &+\left(\frac{dm}{ds}\right)^{-2}\begin{bmatrix} M \\ s \end{bmatrix}_\alpha\end{aligned}\right]
\end{aligned}\right\} = 0
$$